# CALCULATION OF FLOW AND BED DEFORMATION WITH A GENERAL NON-ORTHOGONAL COORDINATE SYSTEM


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SUMMARY: A two dimensional numerical model to calculate the flow and bed deformation in a general non-orthogonal coordinate system is introduced. The model is used to calculate the flow in a meandering channel with compound cross section, and the bed deformation in a channel with abrupt increases and decreases in width. The calculated results compare well with observed values, supporting the validity of the new model.


## Introduction

Several studies have been made to evaluate flow and bed variations in alluvial channels (Engelund 1974; Smith and Mclean 1984; Ikeda et al. 1987; Struiksma et al. 1985; Colombini et al. 1987; Nelson 1988; Shimizu and Itakura 1989; Shimizu et al. 1990). Most previous work was based on co-orthogonal coordinate systems like cartesian, cylindrical, or other curvilinear coordinate systems, applied to the channels with uniform or quasi-uniform width. However, there are often width variations in the downstream direction of rivers of natural geometry, and also complicated channel shapes due to artificial structures in rivers passing through urban areas or in artificial channels. Modeling of these rivers with co-orthogonal coordinate systems is not convenient because of the difficulty of schematization and establishing the appropriate boundary conditions.

This paper attempts to apply a two-dimensional model with a general coordinate system to calculate the flow field and bed deformation in rivers. The flow and bed deformation equations in the curvilinear coordinate system employed in the model proposed by Shimizu and Itakura(1989) are transformed into a general coordinate system. Here, the two coordinates are not necessarily co-orthogonal, and may be applied to any shape of channel geometry. The boundary conditions along the lateral boundary are easily to be set by introducing contravariant components of flow and sediment flux. The transformed equations are solved numerically by a finite difference method.

The model is applied to calculate the flow field of channels with compound cross sections where low-water channels and flood plains of different curvatures, and river bed deformation in channels with abrupt increases and decreases of width. The results of these applications compare favorably with observed values.

## Flow Equations

Assuming that the vertical component of velocity is negligible and that the pressure is hydrostatic, the two-dimensional equation for water depth and depth-averaged velocities, using a cartesian coordinate system, can be described as follows (Shimizu and Itakura 1989):

$$\frac{\partial(hu^x)}{\partial x}+\frac{\partial(hu^y)}{\partial y}=0 \tag{1}$$

$$\frac{\partial(u^x)^2}{\partial x}+\frac{\partial(u^xu^y)}{\partial y}=-g\frac{\partial H}{\partial x}-\frac{\tau_{bx}}{\rho h}+\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial u^x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon\frac{\partial u^x}{\partial y}\right) \tag{2}$$

$$\frac{\partial(u^xu^y)}{\partial x}+\frac{\partial(u^y)^2}{\partial y}=-g\frac{\partial H}{\partial y}-\frac{\tau_{by}}{\rho h}+\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial u^y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon\frac{\partial u^y}{\partial y}\right) \tag{3}$$

in which $x$ and $y$ = a system of cartesian coordinates; $u^x$ and $u^y$ = $x$ and $y$ components of the velocity vector; $H$ = water surface elevation; $h$ = water depth; $g$ = acceleration of gravity; $\rho$ = density of fluid; $\varepsilon$ = eddy viscosity; and $\tau_{bx}$ and $\tau_{by}$ = the boundary shear stress components along the channel bed in $x$ and $y$ direction. The $\tau_{bx}$, $\tau_{by}$, and $\varepsilon$ terms in the momentum equations can be written as

$$\tau_{bx} = \rho C_f \sqrt{(u^x)^2 + (u^y)^2}, \qquad \tau_{by} = \rho C_f \sqrt{(u^x)^2 + (u^y)^2}, \qquad \varepsilon = \frac{\kappa}{6} u_* h \tag{4}$$

in which $C_f$ = bed friction coefficient; $u_*$ = shear velocity = $C_f\sqrt{(u_x)^2 + (u_y)^2}$; and $\kappa$ = Von Karman's constant. The flow equation in a $(x,\ y)$ coordinate system can be transformed into a general coordinate system $(\psi,\phi)$ using the following relationships:

$$\frac{\partial}{\partial x} = \psi_x \frac{\partial}{\partial \psi} + \phi_x \frac{\partial}{\partial \phi}, \qquad \frac{\partial}{\partial y} = \psi_y \frac{\partial}{\partial \psi} + \phi_y \frac{\partial}{\partial \phi} \tag{5}$$

$$u^x = \frac{1}{J}\left(\phi_y u^\psi - \psi_y u^\phi\right), \qquad u^y = \frac{1}{J}\left(-\phi_x u^\psi + \psi_x u^\phi\right) \tag{6}$$

$$u^\psi = u^x \psi_x + u^y \psi_y, \qquad u^\phi = u^x \phi_x + u^y \phi_y \tag{7}$$

in which $\psi_x = \partial\psi/\partial x$; $\psi_y = \partial\psi/\partial y$; $\phi_x = \partial\phi/\partial x$; $\phi_y = \partial\phi/\partial y$; $u^\psi$ and $u^\phi$ = contravariant velocity components in $\psi$; and $\phi$-direction, respectively. The transformed equations are

$$\frac{\partial}{\partial \psi}\left(\frac{hu^\psi}{J}\right) + \frac{\partial}{\partial \phi}\left(\frac{hu^\phi}{J}\right) = 0 \tag{8}$$

$$\begin{aligned}
&\psi_x \left\{ \frac{\partial}{\partial \psi}\left[\frac{1}{J^2}\left(\phi_y u^\psi - \psi_y u^\phi\right) u^\psi\right] + \frac{\partial}{\partial \phi}\left[\frac{1}{J^2}\left(\phi_y u^\psi - \psi_y u^\phi\right) u^\phi\right] \right\} \\
&+\psi_y \left\{ \frac{\partial}{\partial \psi}\left[\frac{1}{J^2}\left(-\phi_x u^\psi + \psi_x u^\phi\right) u^\psi\right] + \frac{\partial}{\partial \phi}\left[\frac{1}{J^2}\left(-\phi_x u^\psi + \psi_x u^\phi\right) u^\phi\right] \right\} \\
&= -\frac{g}{J}\left[(\psi_x^2 + \psi_y^2)\frac{\partial H}{\partial \psi} + (\psi_x\phi_x + \psi_y\phi_y)\frac{\partial H}{\partial \phi}\right] \\
&- \frac{\rho C_f}{J^2} u^\psi \sqrt{\left(\phi_y u^\psi - \psi_y u^\phi\right)^2 + \left(-\phi_x u^\psi + \psi_x u^\phi\right)^2} + D^\psi
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\phi_x \left\{ \frac{\partial}{\partial \psi}\left[\frac{1}{J^2}\left(\phi_y u^\psi - \psi_y u^\phi\right) u^\psi\right] + \frac{\partial}{\partial \phi}\left[\frac{1}{J^2}\left(\phi_y u^\psi - \psi_y u^\phi\right) u^\phi\right] \right\} \\
&+\phi_y \left\{ \frac{\partial}{\partial \psi}\left[\frac{1}{J^2}\left(-\phi_x u^\psi + \psi_x u^\phi\right) u^\psi\right] + \frac{\partial}{\partial \phi}\left[\frac{1}{J^2}\left(-\phi_x u^\psi + \psi_x u^\phi\right) u^\phi\right] \right\} \\
&= -\frac{g}{J}\left[(\psi_x\phi_x + \psi_y\phi_y)\frac{\partial H}{\partial \psi} + (\phi_x^2 + \phi_y^2)\frac{\partial H}{\partial \phi}\right] \\
&- \frac{\rho C_f}{J^2} u^\phi \sqrt{\left(\phi_y u^\psi - \psi_y u^\phi\right)^2 + \left(-\phi_x u^\psi + \psi_x u^\phi\right)^2} + D^\phi
\end{aligned} \tag{10}$$

where $J$ = Jacobian of the coordinate transformation = $\phi_y\psi_x - \psi_y\phi_x$; and $D^\psi$, $D^\phi$ = momentum diffusion terms in $\psi$- and $\phi$-equations, respectively, as described in the Appendix.

## Sediment Transport Equations

The continuity equation for two-dimensional bed-load transport in a general coordinate system is

$$\frac{1-\lambda}{J}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \psi}\left(\frac{q^{\psi}}{J}\right) + \frac{\partial}{\partial \phi}\left(\frac{q^{\phi}}{J}\right) = 0 \tag{11}$$

in which $\eta$ = bed elevation = $H - h$; $t$ = time; $\lambda$ = porosity of bed material; and $q^{\phi}$ and $q^{\psi}$ = contravariant components of the volume rate of bed transport per unit width in $\psi$- and $\phi$-direction, respectively. The $q^{\psi}$ and $q^{\phi}$ terms can be written using the direction of the stream line $s$ and its normal line $n$, as

$$q^{\psi} = \frac{\partial \psi}{\partial s}q^{s} + \frac{\partial \psi}{\partial n}q^{n} = \left(\psi_x\frac{\partial x}{\partial s} + \psi_y\frac{\partial y}{\partial s}\right)q^{s} + \left(\psi_x\frac{\partial x}{\partial n} + \psi_y\frac{\partial y}{\partial n}\right)q^{n} \tag{12}$$

$$q^{\phi} = \frac{\partial \phi}{\partial s}q^{s} + \frac{\partial \phi}{\partial n}q^{n} = \left(\phi_x\frac{\partial x}{\partial s} + \phi_y\frac{\partial y}{\partial s}\right)q^{s} + \left(\phi_x\frac{\partial x}{\partial n} + \phi_y\frac{\partial y}{\partial n}\right)q^{n} \tag{13}$$

The partial derivatives of $x$ and $y$ in $s$- and $n$-directions can be described, considering the local direction of the flow, as

$$\frac{\partial x}{\partial s} = \frac{u^x}{V}, \qquad \frac{\partial x}{\partial n} = -\frac{u^y}{V}, \qquad \frac{\partial y}{\partial s} = \frac{u^y}{V}, \qquad \frac{\partial y}{\partial n} = \frac{u^x}{V} \tag{14}$$

in which $V = \sqrt{(u^x)^2 + (u^y)^2}$. Substituting Eq. 14 into Eqs. 12 and 13, $q^{\psi}$ and $q^{\phi}$ can be rewritten as follows:

$$q^{\psi} = \left(\psi_x\frac{u^x}{V} + \psi_y\frac{u^y}{V}\right)q^{s} + \left(-\psi_x\frac{u^y}{V} + \psi_y\frac{u^x}{V}\right)q^{n} = \frac{1}{V}\left(u^{\psi}q^{s} - Ju_{\phi}q^{n}\right) \tag{15}$$

$$q^{\phi} = \left(\phi_x\frac{u^x}{V} + \phi_y\frac{u^y}{V}\right)q^{s} + \left(-\phi_x\frac{u^y}{V} + \phi_y\frac{u^x}{V}\right)q^{n} = \frac{1}{V}\left(u^{\phi}q^{s} - Ju_{\psi}q^{n}\right) \tag{16}$$

where $u_{\psi}$ and $u_{\phi}$ = covariants of the depth-averaged velocity components in $\psi$- and $\phi$- direction, respectively, described as

$$u_{\psi} = \frac{1}{J}(\phi_y u^x - \phi_x u^y), \qquad u_{\phi} = \frac{1}{J}(-\psi_y u^x + \psi_x u^y) \tag{17}$$

$q^s$ may be calculated from any bed-load transport equation. For convenience, the Mayer-Peter-Muller formula is adopted:

$$q^{s} = 8\sqrt{\left(\frac{\rho_s - 1}{\rho}\right)gd^3}(\tau_* - \tau_{*c})^{3/2} \tag{18}$$

where $\rho$ = specific density of sediment; $d$ = particle size of sediment; $\tau_*$ = nondimensional bed stress = $u_*^2/[(\rho_s - 1)/\rho g d]$; and $\tau_{*c}$ = critical shear stress calculated by Iwagaki's (1956) formula.

When the $s$-direction is identical to the stream line, the depth-averaged velocity component in $n$-direction becomes zero, and $q^n$ due to the depth-averaged flow also becomes zero. However, in curved flow, secondary flow (spiral flow in transverse direction) induced by the centrifugal force acting on the curved stream line is developed and $q^n$ driven by the secondary flow is produced, which is very important factor to predict river bed topography in curved flow. Further important factor is the effect of gravity on the sand particles in the transverse slope of the bed surface. Hasegawa's(1984) formula for $q^n$ is adopted to account for these effects:

$$q^n = q^s \left( \frac{h}{r} N_* - \sqrt{\frac{\tau_*}{\mu_s \mu_k}} \frac{\partial \eta}{\partial n} \right) \tag{19}$$

in which $r$ = radius of curvature of stream line; $N_*$ = a coefficient to describe the intensity of secondary flow; $\mu_s$ = static friction factor(1.0); and $\mu_k$ = kinetic friction factor of sediment particles on the river bed (0.45). Eq. 19 has been tested successfully by Shimizu and Itakura (1989) for various channel geometries. $N_* = 7.0$ is adopted as proposed by Engelund (1974), and $r$ and $\partial\eta/\partial n$ can be calculated in ($\psi$,$\phi$) coordinate system as

$$\frac{1}{r} = \frac{1}{V^3}\left[ u^\psi \left( \frac{\partial u^y}{\partial \psi} u^x - \frac{\partial u^x}{\partial \psi} u^y \right) + u^\phi \left( \frac{\partial u^y}{\partial \phi} u^x - \frac{\partial u^x}{\partial \phi} u^y \right) \right], \quad \frac{\partial \eta}{\partial n} = \frac{J}{V}\left( u_\psi \frac{\partial \eta}{\partial \phi} - u_\phi \frac{\partial \eta}{\partial \psi} \right) \tag{20}$$

## Computations

The equations described in the preceding section are solved with a finite difference method on a channel-fitted computational grid. The grid need not be co-orthogonal, and can be applied to any channel geometry. Previous to the calculations of flow and river bed variations, the values of $\psi_x$, $\phi_x$, $\psi_y$, $\psi_y$, and $J$ are calculated at each point of the ($\psi$,$\phi$) grid. The contravariant velocity components, $u^\psi$ $u^\phi$, and water depth $h$, are calculated from Eqs. 9, 10, and 8, respectively: the sediment transport rates, $q^s$ and $q^n$, are calculated with Eqs. 18 and 19, the bed elevation $\eta$ is calculated with Eq. 11, while $q^\psi$ and $q^\phi$ are calculated by Eqs. 15 and 16.

When the flow is subcritical, $u^\psi$, $u^\phi$ at the upstream end, and $H$ at the downstream end are given as the boundary conditions. The contravariant components in the direction across side boundaries like side walls or banks are zero. Further details of the numerical computations are described elsewhere (Shimizu 1990).

## Application of the Flow Model

This section presents the results of the flow model with the experimental results obtained by Nishimura (1984) in a flume with compound cross section. The experimental flume has a meandering low water channel with the flood plain connecting to straight side walls. A typical cross section of the experimental flume is shown in Fig. 1. The experiment was conducted with the conditions described in Table 1, and the depth averaged velocity vectors were measured.

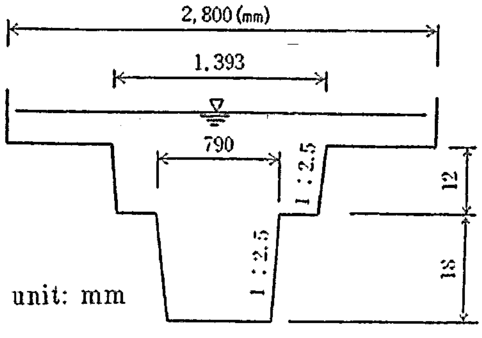


Fig. 1 Cross section of the flume

Table-1 Experimental condition

| | |
|---|---|
| discharge ($\ell$/s) | 8.33 |
| slope | 1/4,000 |
| average depth (cm) | 4.3 |
| channel width (cm) | 280 |
| channel bed | smooth |

Numerical computations were carried out with the conditions in the experiment. Assuming hydraulically smooth flow, and the logarithmic profile of the velocity distribution, $C_f$ is calculated from,

$$C_f = \left[ \frac{\kappa}{\ln \frac{9u_* h}{\nu} - 1} \right]^2 \tag{21}$$

in which $\nu$ = viscosity of water. Since the curvature of the low water channel and the side walls are different, the computational grid is set as shown in Fig. 2.

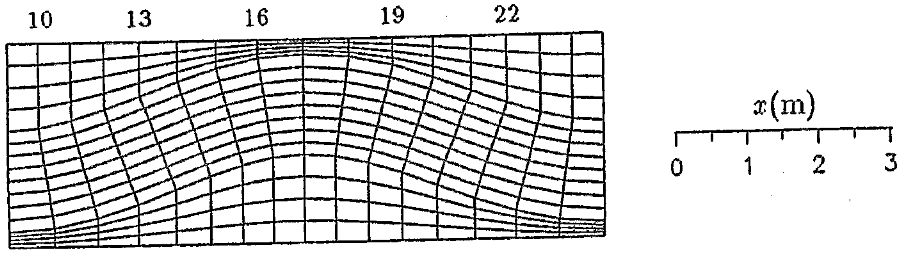


Fig. 2 Computational grid for Nishimura's (1984) flume

Fig. 3 shows the (a)observed and (b)calculated depth-averaged velocity vectors. In Fig. 4, the predicted depth-averaged velocity components, $u^x$ and $u^y$, are compared with the measured ones, in which circles and solid lines are the observed and predicted, respectively. There is a good agreement between the measured and calculated values in Fig. 3 and Fig. 4, verifying the adequacy of the flow model.

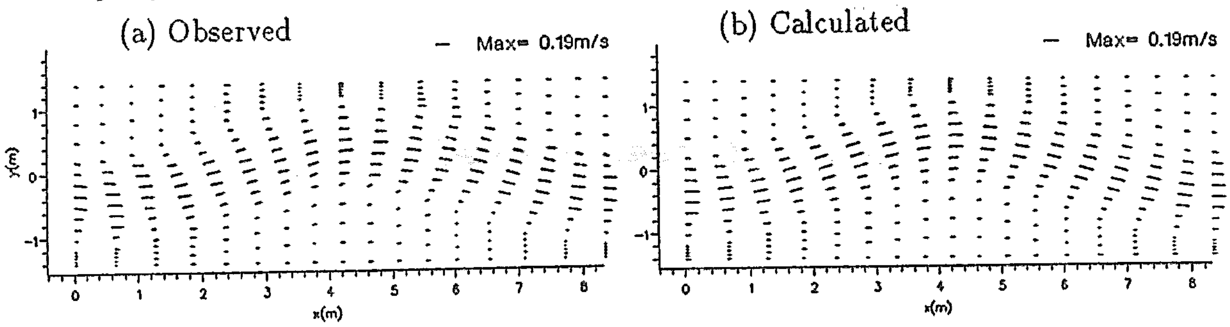


Fig. 3 Observed and calculated velocity vectors.

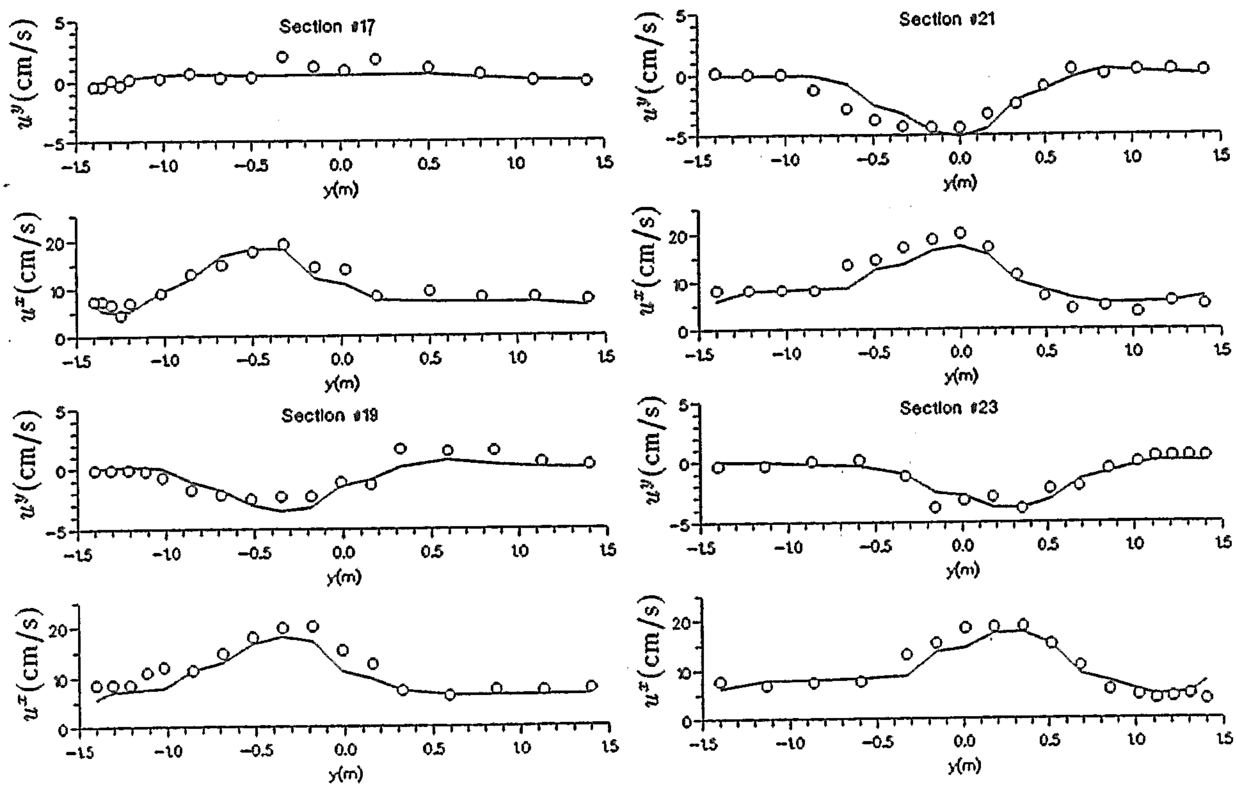


Fig. 4 Cross-sectional distribution of depth-averaged velocity

## Application of the Bed Variation Model

The flow model was coupled with the sediment transport and bed variation model, and tested with data from the experiments with a movable bed conducted by Itakura et al.(1988). The plane geometry of the experiment is shown in Fig. 5, which includes a straight narrow 0.2 m wide passage connected to a 1 m width entrance and a exit. The flow data of the experiment is summarized in Table-2.

Table-2 Experimental condition

| | |
|---|---|
| Flow rate ($\ell$/s) | 20.6 |
| Channel length (m) | 20 |
| Channel width (m) | 1.0 |
| Size of bed material (mm) | 0.94 |
| Initial bed slope | 1/330 |
| Time to reach equilibrium (min.) | 135 |

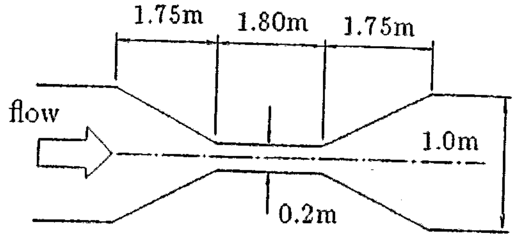


Fig. 5 Geometry of the flume

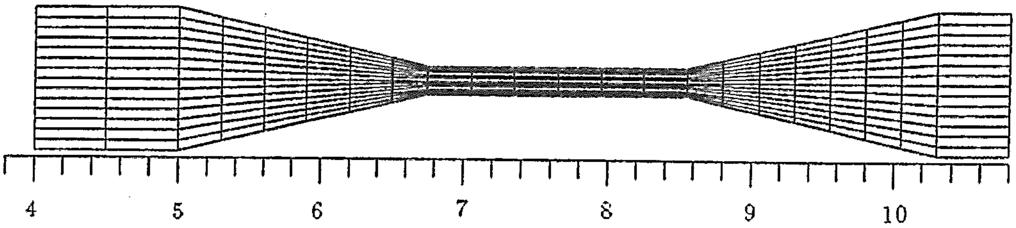


Fig. 6 Computational grid for the experiment by Itakura et al.(1988)

The experiment was started with flat-bed conditions and continued for 135 min. Fig. 6 shows the computational grid for the calculation. The transverse direction of the flow is divided into 15 uniform grids and thus the coordinate system used here is partly non-orthogonal. The numerical computations are carried out for the same situation as the experiment. In the computation, the dynamic equilibrium condition of the bed-load transport is assumed at the upstream end, and $\partial q^s/\partial s = 0$ at the upstream end is given as a boundary condition. The bed friction coefficient, $C_f$, is calculated using Kishi and Kuroki's (1972) formula for flat bed condition as,

$$C_f = 0.22\left(\frac{d}{h}\right)^{1/3} \tag{22}$$

Fig. 7(a) shows the bed configuration, and each contour line denotes the deviation of the bed elevation from the initial state. At the entrance to the narrow passage ($X = 5 \sim 6$ m), scours along the side walls are observed, suggesting the existence of a secondary flow produced by the curved stream. At the stage of the narrow path ($X = 6.5 \sim 7$ m), the central part of the cross section is eroded. In the narrow path ($X = 7.5 \sim 8$ m), the bed is uniformly eroded by the strong acceleration of the flow. At the exit from the narrow path ($X = 9 \sim 10$ m), both sides are eroded, after which it approaches the initial flat bed at the exit of the narrow path ($X = 10$ m).

Fig. 7(b) shows the calculated bed configuration. The characteristics of the observed bed deformation is favorably predicted by the present model. The biggest difference of the grid size across the stream is 5 times, which produces a strong non-orthogonal effect, however, the calculations show good agreement with the observed results, verifying the validity of the model.

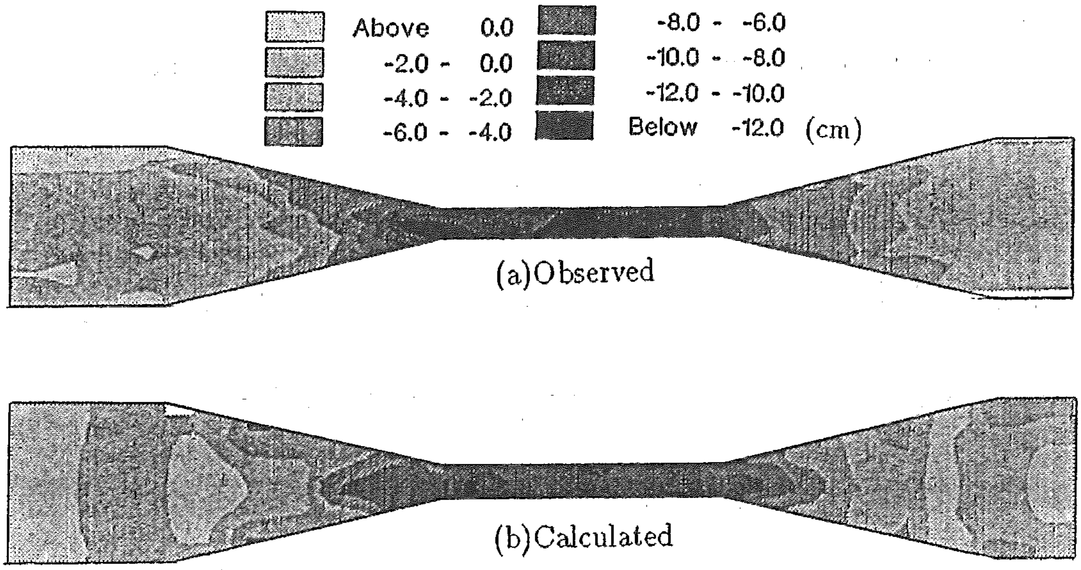


Fig. 7 Bed configuration in the experiment by Itakura et al. (1988).
(a) Observed and (b) Predicted.

## Conclusions

Two-dimensional equations for flow field and bed load transportation in a general non-orthogonal coordinate system are presented. A new numerical model was developed to calculated two-dimensional flow and bed deformation, where the effect of the secondary flow in curved flow was taken into account. The model was applied to flow field in a compound meandering channel, and to bed variations in a channel with abrupt changes in width. The results showed good agreement with the experimental results. The geometries of the examples presented in this paper were relatively simple, however, they included important features of a general coordinate system. The proposed model is very convenient and would be useful in river planning works.

## References

Colombini, M., Seminara, G., and Tubino, M. (1987). "Finite-amplitude alternate bars." *J. Fluid Mech.* , 181, 213–232.

Engelund, F. (1974). "Flow and bed topography in channel bends." *J. Hyd. Div.*, ASCE, 100(11), 1631–1648.

Hasegawa, K. (1984). "Hydraulic research on planimetric forms, bed topographies and flow in alluvial rivers." Ph. D. Dissertation, Hokkaido Univ., Sapporo, Japan, 1–184 (in Japanese).

Ikeda, S., Yamasaka, M., and Chiyoda, M. (1987). "Bed topography and sorting in bends." *J. Hyd. Div.*, ASCE, 113(2), 190–206.

Itakura, T., Kishi, T., Mori, A., and Kuroki, M. (1988). "A study on the flow and bed deformation in a channel with abrupt changes of the width." Fundamental Studies of Functions of Rivers, Hokkaido Development Bureau, 51-70 (in Japanese).

Iwagaki, Y. (1956). "Hydrodynamical study on critical tractive force." *Proc. JSCE* , 41, 1-21 (in Japanese).

Kishi, T., and Kuroki, M. (1972). "Bed forms and resistance to flow in erodible-bed channels(1)." Bulletin of the Faculty of Engrg., Hokkaido Univ., 67, 1–23 (in Japanese).

Nelson., J. M. (1988). "Mechanics of flow and sediment transport over nonuniform erodible beds." Ph.D. Dissertation, Univ. of Wa., 1-227.

Smith, J., and Mclean, S. (1984). "A model for flow in meandering streams." *Water Resources Research*, 20(9), 1301-1315.

Shimizu, Y., and Itakura, T. (1989). "Calculation of bed variation in alluvial channels." *J. Hyd. Engrg.*, ASCE, 115(3), 367-384.

Shimizu, Y., Yamaguchi, H., and Itakura, T. (1990). "Three-dimensional computation of flow and bed deformation." *J. Hyd. Engrg.*, ASCE, 116(9), 1090–1108.

Struiksma, N., Olesen, K.W., Flokstra, C., and DeVriend, H.J. (1985). "Bed deformation in curved alluvial channels." *J. Hyd. Res.*, IAHR, 23(1),57-79.

## Appendix

Momentum diffusion terms in Eqs. 9 and 10, $D^\psi$ and $D^\phi$, are described as follows:

$$
\begin{aligned}
D^\psi = \\
&\frac{\psi_x}{J}\left\{\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right.\\
&\left.+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right\}\\
&+\frac{\psi_x}{J}\left\{\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right.\\
&\left.+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right\}\\
&+\frac{\psi_y}{J}\left\{-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right.\\
&\left.-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right\}\\
&+\frac{\psi_y}{J}\left\{-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right.\\
&\left.-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
D^\phi = \\
&\frac{\phi_x}{J}\left\{\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right.\\
&\left.+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right\}\\
&+\frac{\phi_x}{J}\left\{\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right.\\
&\left.+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_y u^\phi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_y u^\psi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_y u^\phi}{J}\right)\right]\right\}\\
&+\frac{\phi_y}{J}\left\{-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_x\frac{\partial}{\partial\psi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right.\\
&\left.-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\psi_x\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_x\frac{\partial}{\partial\phi}\left[\varepsilon\phi_x\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right\}\\
&+\frac{\phi_y}{J}\left\{-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\psi_y\frac{\partial}{\partial\psi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right.\\
&\left.-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\psi_y\frac{\partial}{\partial\psi}\left(\frac{\psi_x u^\phi}{J}\right)\right]-\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\phi_x u^\psi}{J}\right)\right]+\phi_y\frac{\partial}{\partial\phi}\left[\varepsilon\phi_y\frac{\partial}{\partial\phi}\left(\frac{\psi_x u^\phi}{J}\right)\right]\right\}
\end{aligned}
$$